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A Lie algebraic approach to Novikov algebras

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Abstract

Novikov algebras were introduced in connection with Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus. The commutator of a Novikov algebra is a Lie algebra. Thus it is useful to relate the study of Novikov algebras to the theory of Lie algebras. In this paper, we will try to realize Novikov algebras through a Lie algebraic approach. Such a realization could be important in physics and geometry. We find that all transitive Novikov algebras in dimension ≤ 3 can be realized as the Novikov algebras obtained through Lie algebras and their compatible linear (global) deformations.

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1. Introduction

Hamiltonian operators have close relations with certain algebraic structures [1–8]. Gel'fand and Diki [1,2] introduced a formal variational calculus and found certain interesting Poisson structures when they studied Hamiltonian systems related to certain nonlinear partial differential equations, such as KdV equations. In Ref. [3], more connections between Hamiltonian operators and certain algebraic structures were found. Dubrovin et al. [4–6] studied similar Poisson structures from another point of view. One of the algebraic

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structures appearing in Refs. [3,6], which is called a "Novikov algebra" by Osborn and Xu [9–13], was introduced in connection with Poisson brackets of hydrodynamic type:

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + \sum_{k=1}^{N} u_x^k b_k^{ij}(u(x))\delta(x-y). \tag{1.1}$$

A Novikov algebra A is a vector space over a field $\mathbf{F}$ with a bilinear product $(x, y) \to xy$ satisfying

$$(x_1, x_2, x_3) = (x_2, x_1, x_3) \tag{1.2}$$

and

$$(x_1x_2)x_3 = (x_1x_3)x_2 \tag{1.3}$$

for $x_1, x_2, x_3 \in A$, where

$$(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3). \tag{1.4}$$

Novikov algebras are a special class of left-symmetric algebras which only satisfy Eq. (1.2). Left-symmetric algebras are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [14–17].

The commutator of a Novikov algebra (or a left-symmetric algebra) A

$$[x, y] = xy - yx \tag{1.5}$$

defines a (sub-adjacent) Lie algebra $\mathcal{G} = \mathcal{G}(A)$. Let L_x, R_x denote the left and right multiplication, respectively, i.e. $L_x(y) = xy$, $R_x(y) = yx\, \forall x, y \in A$. Then for a Novikov algebra, the left multiplication operators form a Lie algebra and the right multiplication operators are commutative.

Zel'manov [18] gave a fundamental structure theory of finite-dimensional Novikov algebras over an algebraically closed field with characteristic 0. A Novikov algebra is called right-nilpotent or transitive if every R_x is nilpotent. Then by Eq. (1.3), a finite-dimensional Novikov algebra A contains the (unique) largest transitive ideal $N(A)$ (is called the radical of A) and the quotient algebra $A/N(A)$ is a direct sum of fields. The transitivity corresponds to the completeness of the affine manifolds in geometry [14,15].

Due to the non-associativity, there is not a suitable representation theory for Novikov algebras. In particular, it is quite difficult to study the non-associative Novikov algebras [19]. Therefore, to find certain realization of Novikov algebras can be regarded as the first step. Here, the so-called realization means that we should use some well-known structures to realize Novikov algebras. The first important class of Novikov algebras are given through the commutative associative algebras as follows [20,21]. Let $(A, \cdot)$ be a commutative associative algebra, and D be its derivation. Then the new product

$$a *_x b = a \cdot Db + x \cdot a \cdot b \tag{1.6}$$

makes $(A, *_x)$ to become a Novikov algebra for $x = 0$ by Gel'fand [3], for $x \in \mathbf{F}$ by Filipov [22] and for a fixed element $x \in A$ by Xu [13]. In Ref. [20], we show that the algebra $(A, *) = (A, *_0)$ given by Gel'fand is transitive. We also construct a deformation theory and the Novikov algebras given by Filipov and Xu are the special compatible linear (global) deformations of ones given by Gel'fand. Moreover, we prove that all transitive Novikov algebras in dimension ≤3 can be realized as the algebras defined by Gel'fand and their compatible linear (global) deformations. We conjecture that this conclusion can be extended to higher dimensions. We also extend such a realization theory to the non-transitive Novikov algebras [21]. Such a realization theory has been used to discuss the bilinear forms on Novikov algebras [23] and the derivations of Novikov algebras [24].

However, it is a little insufficient for such a realization, in particular when we discuss some properties of Novikov algebras related to their sub-adjacent Lie algebras. The relation between Novikov algebras and Lie algebras is the key to study the geometry of Novikov algebras which obviously plays an important role in the theory of Novikov algebras. And certain application of Novikov algebras in physics is also given through Lie algebras [6]. But, it is quite a difference between commutative associative algebras and Lie algebras. On the other hand, Lie algebras are a class of non-associative algebras which have been studied for a long time. Any Lie algebraic approach to realize Novikov algebras will be useful to understand the geometry and the physics related to Novikov algebras and may be much closer to the non-associativity of Novikov algebras. We extend the idea of Gel'fand as follows. Let $(A, [,])$ be a Lie algebra, f be a linear transformation on it. Then by choosing f suitably, we can define a Novikov algebra as

$$x * y = [f(x), y] \quad \forall x, y \in A. \tag{1.7}$$

In this paper, we commence to study such a realization. The paper is organized as follows. In Section 2, we study the so-called Novikov interior derivation algebras, which are a class of Novikov algebras realized by Eq. (1.7) with an additional condition that the Lie algebras are just their sub-adjacent Lie algebras. These Novikov algebras have an interesting geometric background. In Section 3, the article to be self-contained, we briefly introduce the deformation theory of Novikov algebras given in Ref. [20], and we give the realization of all transitive Novikov algebras in dimensions 2 and 3 through the Lie algebraic approach. In Section 4, we briefly discuss another Lie algebraic approach to Novikov algebras. In Section 5, we give some conclusions based on the discussion in the previous sections.

2. Novikov interior derivation algebras

Let A be a Lie algebra and f be a linear transformation on it. Then Eq. (1.7) defines a Novikov algebra if and only if f satisfies

$$f([f(x), y] + [x, f(y)]) - [f(x), f(y)] \in C(A), \tag{2.1}$$

$$[f([f(x), y]), z] = [f([f(x), z]), y] \tag{2.2}$$

for any $x, y, z \in A$ and $C(A) = \{a \in A | [a, b] = 0 \, \forall \, b \in A\}$ is the center of the Lie algebra A. In fact, Eq. (2.1) corresponds to Eq. (1.2) and Eq. (2.2) follows from Eq. (1.3).

Obviously, when A is Abelian or f is zero, the resultant Novikov algebra is the trivial Novikov algebra, i.e. all products are zero. Thus, we mainly discuss the cases when A is not Abelian and f is not zero.

In particular, there are a special class of Novikov algebras which are very interesting and important. The sub-adjacent Lie algebra of the Novikov algebra defined as above is just the former Lie algebra. That means, besides Eqs. (2.1) and (2.2), f should satisfy an additional condition:

$$[x, y] = [f(x), y] + [x, f(y)] \quad \forall x, y \in A. \tag{2.3}$$

Thus, Eq. (2.1) becomes

$$f([x, y]) - [f(x), f(y)] \in C(A). \tag{2.4}$$

Furthermore, from Eqs. (2.3) and (2.4) and the Jacobi identity, we can obtain

$$[f([f(y), x]), z] = [f([f(z), y]), x] \quad \forall x, y, z \in A. \tag{2.5}$$

Thus, combining Eqs. (2.2), (2.3) and (2.5) together, we can obtain

$$[f([f(y), x]), z] - [f([f(x), y]), z] = [f([x, y]), z] = 0 \quad \forall x, y, z \in A, \tag{2.6}$$

i.e.:

$$f([x, y]) \in C(A). \tag{2.7}$$

Through Eq. (2.4), we have

$$[f(x), f(y)] \in C(A). \tag{2.8}$$

Moreover, the above process is sufficient and necessary. It is easy to find that Eqs. (2.1)–(2.3) hold if and only if Eq. (2.3) and Eqs. (2.7) and (2.8) hold, i.e. f satisfies

$$[x, y] = [f(x), y] + [x, f(y)], \qquad f([x, y]) \in C(A),$$
$$[f(x), f(y)] \in C(A) \quad \forall x, y \in A. \tag{2.9}$$

The left multiplication operator L_x for above Novikov algebra is just the linear transformation $\operatorname{ad} f(x)$ for its sub-adjacent Lie algebra, where ad is the adjoint operator which satisfies $\operatorname{ad} x(y) = [x, y] \forall x, y \in A$. Therefore, any Novikov algebra A defined by Eq. (1.7) with f satisfying Eqs. (2.1)–(2.3) (i.e. Eq. (2.9)) has the property: the left multiplication operator L_x or the right multiplication operator $R_x = \operatorname{ad} x - L_x$ is an interior derivation of its sub-adjacent Lie algebra. On the contrary, for a Novikov algebra with such a property, it is easy to show there exists a linear transformation f such that $L_x = \operatorname{ad} f(x)$ and f satisfies Eq. (2.9). Such a Novikov algebra is called a Novikov interior derivation algebra.

In fact, the above structure was first studied in Ref. [25] for a left-symmetric algebra. There is a good geometry behind it. Let G be a Lie group with Lie algebra $\mathcal{G}$, and let $\mathrm{Int}(\mathcal{G})$ denote the group of interior automorphisms of the Lie algebra $\mathcal{G}$. The local interior automorphism structure of G is the principal fiber bundle of frames of G obtained by the extension to $\mathrm{Int}(\mathcal{G})$ of a left-invariant parallelism of G. Its fibers are unique up to a right translation in G's frame bundle $R(G)$. Then the interior derivation algebras just correspond to the left-invariant locally flat connections (defined by the algebras themselves) adapted to the structures defined above. Hence the relationship between these interior derivation algebras and their sub-adjacent Lie algebras can ensure any consequences for the group structure. For more details see Ref. [25]. We also discuss the so-called Novikov derivation algebras which are the Novikov algebra structures adapted to the general automorphism structure of a Lie group in Ref. [26].

From Eq. (2.9) directly or through the discussion in Ref. [25], we can obtain the following properties of Novikov interior derivation algebras:

(a) The sub-adjacent Lie algebra of a Novikov interior derivation algebra A can be decomposed into a direct sum of three vector subspaces A_1, A_2, A_3, which satisfy the following conditions:

$$[A_1, A_1] = [A_2, A_2] = [A_1, A_3] = [A_2, A_3] = 0, [A_1, A_2] \subset A_1 + C(A), [A_3, A_3] \subset C(A). \quad (2.10)$$

In particular, A must be 2-solvable, i.e. the derived Lie ideal $D(A) = [A, A]$ is Abelian.

(b) Any Novikov interior derivation algebra is transitive.

(c) Any Novikov interior derivation algebra A is associative if and only if its sub-adjacent Lie algebra A is 2-nilpotent, i.e. the derived Lie ideal $D(A)$ is in the center $C(A)$.

(d) There exists a Novikov interior derivation product on any 2-solvable Lie algebra with trivial center. In fact, from the discussion in Ref. [25], such a Lie algebra A has a decomposition:

$$A = D(A) \oplus C, \quad (2.11)$$

where C is an Abelian Cartan subalgebra of A. Thus the Novikov interior derivation product on A can be defined by

$$L_{a_D} = 0, \qquad L_{a_C} = \mathrm{ad}(a_C), \quad (2.12)$$

where $a_D \in D(A)$, $a_C \in C$.

(e) There are certain kind of 2-solvable Lie algebras with trivial center which have the property that it is sub-adjacent to a unique Novikov interior derivation structure. Such an example can be obtained from Ref. [25]. Let A be an n-dimensional Lie algebra with the product

$$[e_i, e_j] = 0, i, j \geq 2, \qquad [e_1, e_i] = \lambda_i e_i, \quad i \geq 2, \lambda_i \neq 0, \quad \text{the } \lambda_i \text{ being pairwise distinct.} \quad (2.13)$$

The (unique) Novikov interior derivation structure is given by

$$e_1 e_1 = 0, \qquad e_1 e_i = \lambda_i e_i, \qquad e_i e_j = 0, \quad i, j \geq 2. \quad (2.14)$$

At the end of this section, based on the classification of Novikov algebras in dimension ≤ 3 given in Ref. [19], we give the classification of Novikov interior derivation algebras in dimension ≤ 3 and their corresponding realization as follows. Recall that the (form) characteristic matrix of a Novikov algebra is defined as

$$\mathcal{A} = \begin{pmatrix} \sum_{k=1}^{n} c_{11}^{k} e_k & \cdots & \sum_{k=1}^{n} c_{1n}^{k} e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^{n} c_{n1}^{k} e_k & \cdots & \sum_{k=1}^{n} c_{nn}^{k} e_k \end{pmatrix}, \quad (2.15)$$

where $\{e_i\}$ is a basis of A and $e_i e_j = \sum_{k=1}^{n} c_{ij}^k e_k$.

- *One-dimensional Novikov interior derivation algebra*.

$$\text{Lie algebra is Abelian} \Rightarrow \text{trivial Novikov algebra}.$$

- *Two-dimensional Novikov interior derivation algebras*.

$$\text{Lie algebra is Abelian} \Rightarrow \text{trivial Novikov algebra},$$

$$\text{Lie algebra } \langle e_1, e_2 | [e_1, e_2] = e_1 \rangle \quad \text{and}$$
$$\begin{cases} f(e_1) = 0 \\ f(e_2) = e_2 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}.$$

- *Three-dimensional Novikov interior derivation algebras*.

$$\text{Lie algebra is Abelian} \Rightarrow \text{trivial Novikov algebra},$$

$$\text{Lie algebra} \left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_1 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = 0 \end{cases} \right\rangle \quad \text{and}$$
$$\begin{cases} f(e_1) = 0 \\ f(e_2) = -e_2 \\ f(e_3) = -e_3 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & -e_1 & 0 \end{pmatrix},$$

$$\text{Lie algebra} \left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_1 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = 0 \end{cases} \right\rangle \quad \text{and}$$
$$\begin{cases} f(e_1) = 0 \\ f(e_2) = e_3 - e_2 \\ f(e_3) = -le_2 - e_3 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & le_1 \end{pmatrix},$$

$$\text{Lie algebra} \left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_2 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = 0 \end{cases} \right\rangle \quad \text{and}$$
$$\begin{cases} f(e_1) = 0 \\ f(e_2) = 0 \\ f(e_3) = e_3 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix},$$

$$\text{Lie algebra}\left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = le_2 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = -e_1 \end{cases} \right\rangle, \ |l| \leq 1, l \neq 0 \quad \text{and}$$

$$\begin{cases} f(e_1) = 0 \\ f(e_2) = 0 \\ f(e_3) = e_3 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 0 \end{pmatrix}, \ |l| \leq 1, l \neq 0,$$

$$\text{Lie algebra}\left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_1 + e_2 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = -e_1 \end{cases} \right\rangle \quad \text{and}$$

$$\begin{cases} f(e_1) = 0 \\ f(e_2) = 0 \\ f(e_3) = e_3 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix}.$$

3. The deformation theory of Novikov algebras and the realization of transitive Novikov algebras in dimensions 2 and 3

At first, the article to be self-contained, we briefly introduce the deformation theory of Novikov algebras given in Ref. [20]. Let $(A, *)$ be a Novikov algebra, and $g_p : A \times A \to A$ be a bilinear product defined by

$$g_q(a, b) = a * b + qG_1(a, b) + q^2 G_2(a, b) + q^3 G_3(a, b) + \cdots, \tag{3.1}$$

where G_i are bilinear products with $G_0(a, b) = a * b$. We call (A_q, g_q) a deformation of $(A, *)$ if (A_q, g_q) is a family of Novikov algebras. In particular, we call G_1 a linear (global) deformation if the deformation is given by

$$g_q(a, b) = a * b + qG_1(a, b), \tag{3.2}$$

i.e. $G_2 = G_3 = \cdots = 0$. G_1 is a linear (global) deformation if and only if

$$G_1(G_1(a, b), c) - G_1(a, G_1(b, c)) - G_1(G_1(b, a), c) + G_1(b, G_1(a, c)) = 0, \tag{3.3}$$

$$G_1(G_1(a, b), c) = G_1(G_1(a, c), b), \tag{3.4}$$

$$G_1(a, b * c) - G_1(a * b, c) + G_1(b * a, c) - G_1(b, a * c) + a * G_1(b, c) \\ - G_1(a, b) * c + G_1(b, a) * c - b * G_1(a, c) = 0, \tag{3.5}$$

$$G_1(a, b) * c - G_1(a, c) * b + G_1(a * b, c) - G_1(a * c, b) = 0. \tag{3.6}$$

Moreover, G_1 is called a compatible linear (global) deformation if G_1 is symmetric. Any Novikov algebra and its compatible linear (global) deformations have the same sub-adjacent

Lie algebra. A linear (global) deformation is called special if the elements in the family of Novikov algebras (A_q, g_q) defined by Eq. (3.2) are mutually isomorphic for $q \neq 0$.

Next, by direct computation, we can show that any transitive Novikov algebra in dimensions 2 or 3 can be realized as the Novikov algebra defined by Eq. (1.7) with f satisfying Eqs. (2.1) and (2.2) or its (special) compatible linear (global) deformation. Except the Novikov interior derivation algebras given in Section 2, there are the following transitive Novikov algebras which can be obtained from some non-Abelian Lie algebras in dimensions 2 or 3 and some suitable f:

$$\text{Lie algebra } \langle e_1, e_2 | [e_1, e_2] = e_1 \rangle \quad \text{and}$$
$$\begin{cases} f(e_1) = 0 \\ f(e_2) = e_1 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix},$$

$$\text{Lie algebra} \left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_1 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = 0 \end{cases} \right\rangle \quad \text{and}$$
$$\begin{cases} f(e_1) = 0 \\ f(e_2) = 0 \\ f(e_3) = -e_2 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix},$$

$$\text{Lie algebra} \left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_1 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = 0 \end{cases} \right\rangle \quad \text{and}$$
$$\begin{cases} f(e_1) = 0 \\ f(e_2) = e_3 \\ f(e_3) = -e_2 \end{cases} \Rightarrow \text{Novikov algebra} \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}.$$

In fact, the above Novikov algebras also can be regarded as the compatible linear (global) deformations of the trivial Novikov algebras. However, we usually skip these deformations (they are regarded as the trivial cases) since every Novikov algebra can be regarded as a linear (global) deformation of a certain trivial Novikov algebra [20].

According to the classification of transitive Novikov algebras in dimension ≤ 3 [19], there are the following algebras which cannot be realized through Eq. (1.7) directly:

(1) Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & e_2 \end{pmatrix}$$

is a special compatible linear (global) deformation of Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_1 & 0 \end{pmatrix}$$

$$\left(\text{isomorphic to } \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}\right)$$

with

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_2 \end{pmatrix}.$$

(2) Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & le_1 & e_2 \end{pmatrix}, \ l \neq 1$$

is a special compatible linear (global) deformation of Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & le_1 & 0 \end{pmatrix}, \ l \neq 1$$

$$\left(\text{isomorphic to } \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & -\dfrac{(l+1)^2}{(l-1)^2}e_1 \end{pmatrix}\right)$$

with

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_2 \end{pmatrix}.$$

(3) Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & e_2 \end{pmatrix}$$

is a special compatible linear (global) deformation of Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_1 & 0 \end{pmatrix}$$

$$\left(\text{isomorphic to } \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & e_1 \\ 0 & -e_1 & -e_1 \end{pmatrix}\right)$$

with

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_2 \end{pmatrix}.$$

(4) Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}$$

is a special compatible linear (global) deformation of Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}$$

with

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix}.$$

(5) Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & 0 \end{pmatrix}$$

is a special compatible linear (global) deformation of Novikov algebra

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & \frac{1}{2}e_2 & 0 \end{pmatrix}$$

with

$$G_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

4. Another Lie algebraic approach

Obviously we also can define a Novikov algebra through a Lie algebra by

$$x * y = [x, g(y)] \quad \forall x, y \in A, \tag{4.1}$$

where $(A, [,])$ is a Lie algebra and g is a linear transformation on A satisfying

$$[[x, g(y)] + [g(x), y], g(z)] + [y, g([x, g(z)])] - [x, g([y, g(z)])] = 0, \tag{4.2}$$

$$[g(x), g(y)] \in C(A) \tag{4.3}$$

for any $x, y, z \in A$.

When A is a Novikov interior derivation algebra, the realization given by Eq. (1.7) coincides with that given by Eq. (4.1). In fact, because $\mathrm{ad}\, x = L_x - R_x$ and $L_x = \mathrm{ad}\, f(x)$, $R_x = -\mathrm{ad}\, g(x)$, the relation between f and g is given by

$$\mathrm{ad}\, g(x) = \mathrm{ad}(x - f(x)). \tag{4.4}$$

This fact also can be seen from Eq. (2.9) which can be shown to be equivalent to Eqs. (4.2), (4.3) and (2.3).

Although in general, Eqs. (4.2) and (4.3) are not equivalent to Eqs. (2.1) and (2.2), we can still find that all transitive Novikov algebras in dimension ≤ 3 can be realized as the Novikov algebras defined through Eq. (4.1) and their (special) compatible linear (global) deformations. From the discussion in Section 3, we only need discuss the following cases:

$$\text{Lie algebra } \langle e_1, e_2 | [e_1, e_2] = e_1 \rangle \quad \text{and}$$

$$\begin{cases} g(e_1) = 0 \\ g(e_2) = -e_1 \end{cases} \Rightarrow \text{Novikov algebra } \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix},$$

$$\text{Lie algebra } \left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_1 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = 0 \end{cases} \right\rangle \quad \text{and}$$

$$\begin{cases} g(e_1) = 0 \\ g(e_2) = 0 \\ g(e_3) = e_2 \end{cases} \Rightarrow \text{Novikov algebra } \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_1 \end{pmatrix},$$

$$\text{Lie algebra } \left\langle e_1, e_2, e_3 \middle| \begin{cases} [e_3, e_2] = e_1 \\ [e_1, e_2] = 0 \\ [e_1, e_3] = 0 \end{cases} \right\rangle \quad \text{and}$$

$$\begin{cases} g(e_1) = 0 \\ g(e_2) = -e_3 \\ g(e_3) = e_2 \end{cases} \Rightarrow \text{Novikov algebra } \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & e_1 \end{pmatrix}.$$

5. Conclusions and discussion

From the discussion in the previous sections, we have the following conclusions:

(a) All transitive Novikov algebras in dimensions 2 and 3 can be realized as the Novikov algebras defined by Eq. (1.7) with f satisfying Eqs. (2.1) and (2.2) (or by Eq. (4.1) with g satisfying Eqs. (4.2) and (4.3)) and their compatible linear (global) deformations.

(b) In fact, we have seen that the Novikov interior derivation algebras play an important role in the above realization theory. In particular, all non-commutative transitive Novikov algebras in dimensions 2 and 3 can be realized as Novikov interior derivation algebras and their compatible linear (global) deformations. However, such a conclusion cannot extend to higher dimensions, even for 2-solvable Lie algebras in dimension 4. For example, we can show that there does not exist any Novikov interior derivation algebra structure on the 4-dimensional nilpotent Lie algebra [27] given by

$$\langle e_1, e_2, e_2, e_4 | [e_2, e_3] = e_1, [e_3, e_4] = e_2, \text{ other products are zero}\rangle.$$

(c) We would like to point out that, unlike the realization theory given through commutative associative algebras [20,21], the Lie algebraic approach is less useful for the non-transitive Novikov algebras. Even in dimension 2 [19], there are some non-transitive Novikov algebras which cannot be realized as the Novikov algebras obtained through Eq. (1.7) (or Eq. (4.1)) or their compatible linear (global) deformations.

Although there are some limitations for such a Lie algebraic approach to Novikov algebras, it will still be interesting to apply it to physics and geometry.

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